CHAPTER 4

Small-Disturbance Flow over Three-Dimensional Wings: Formulation of the Problem

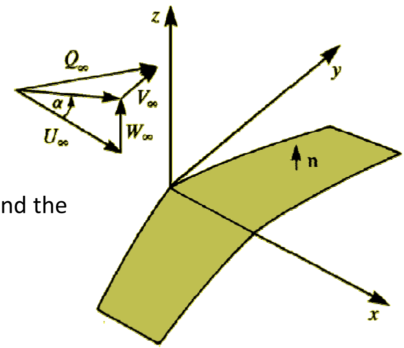
One of the first important applications of potential flow theory was the study of lifting surfaces (wings). Since the boundary conditions on a complex surface can considerably complicate the attempt to solve the problem by analytical means, some simplifying assumptions need to be introduced.
In this chapter assumptions will be applied to the formulation of the 3D thin wing problem and the scene for the singularity solution technique will be set.

4.1 Definition of the Problem

The finite wing is moving at a constant speed in an otherwise undisturbed fluid
The angle of attack α:

$$\alpha = \tan^{-1}\frac{W_\infty}{U_\infty}$$

for the sake of simplicity $V_\infty \equiv 0$



If it is assumed that the fluid surrounding the wing and the wake is inviscid, incompressible, and irrotational

$$\nabla^2\Phi^* = 0 \quad (4.1)$$

BC1: The boundary conditions require that the disturbance induced by the wing will decay far from the wing

$$\lim_{r\to\infty} \nabla\Phi^* = \mathbf{Q}_\infty \quad (4.2)$$

which is automatically fulfilled by the singular solutions such as for the source, doublet, or the vortex elements.

4.1 Definition of the Problem

BC2: Also, the normal component of velocity on the solid boundaries of the wing must be zero.

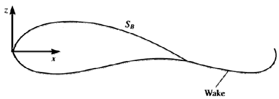
$$\nabla \Phi^* \cdot \mathbf{n} = 0 \qquad (4.3)$$

So, the problem reduces to finding a singularity distribution that will satisfy Eq. (4.3).

The distribution is found → the velocity field (**q**) is known → pressure *p* from the steady-state Bernoulli equation:

$$p_\infty + \frac{\rho}{2} Q_\infty^2 = p + \frac{\rho}{2} q^2 \qquad (4.4)$$

Wake model → Helmholtz theorems
if the wing is modeled by singularity elements that will introduce vorticity these need to be shed into the flow in the form of a wake.



complicated analytical solution for an arbitrary wing shape

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For difficulty of specifying B.C. Eq.(4.3) on a complex shape surface & by shape of a wake.

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some additional simplifying assumptions are made

4.2 The Boundary Condition on the Wing

The wing solid surface be defined as

$$z = \eta(x, y) \qquad (4.5)$$

$$F(x, y, z) \equiv z - \eta(x, y) = 0 \qquad (4.6)$$

The outward normal on the wing upper surface is (from Eq. (2.26)):

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} = \frac{1}{|\nabla F|}\left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1\right) \qquad (4.7)$$

Note: −**n** is outward normal on lower surface

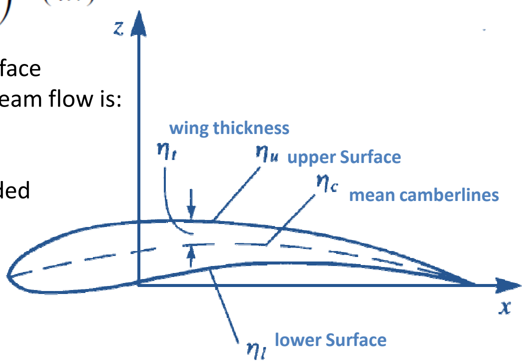
The velocity potential due to the free-stream flow is:

$$\Phi_\infty = U_\infty x + W_\infty z \qquad (4.8)$$

Eq. (4.1) is linear so, solution can be divided into two separate parts:

$$\Phi^* = \Phi + \Phi_\infty \qquad (4.9)$$

↑ Perturbation Velocity Potential



wing with nonzero thickness

4.2 The Boundary Condition on the Wing

Substituting Eq. (4.7) & derivatives of Eqs. (4.8) & (4.9) into B.C. Eq. (4.3)

$$\nabla\Phi^* \cdot \mathbf{n} = \nabla\Phi^* \cdot \frac{\nabla F}{|\nabla F|}$$

$$= \left(\frac{\partial\Phi}{\partial x} + U_\infty, \frac{\partial\Phi}{\partial y}, \frac{\partial\Phi}{\partial z} + W_\infty\right) \cdot \frac{1}{|\nabla F|}\left(-\frac{\partial\eta}{\partial x}, -\frac{\partial\eta}{\partial y}, 1\right) = 0 \quad (4.10)$$

Result: The unknown is the perturbation potential ϕ, which represents the velocity induced by the motion of the wing in a stationary frame of reference.

$$\left.\begin{array}{l}\nabla^2\Phi^* = 0 \\ \Phi^* = \Phi + \Phi_\infty\end{array}\right\} \longrightarrow \nabla^2\Phi = 0 \quad (4.11)$$

B.C. on the wing surface by rearranging ∂ϕ/∂z in Eq.(4.10)

$$\frac{\partial\Phi}{\partial z} = \frac{\partial\eta}{\partial x}\left(U_\infty + \frac{\partial\Phi}{\partial x}\right) + \frac{\partial\eta}{\partial y}\left(\frac{\partial\Phi}{\partial y}\right) - W_\infty \qquad \text{on} \quad z = \eta \qquad (4.12)$$

4.2 The Boundary Condition on the Wing

The classical small-disturbance approximation will allow us to further simplify this B.C. Assume:

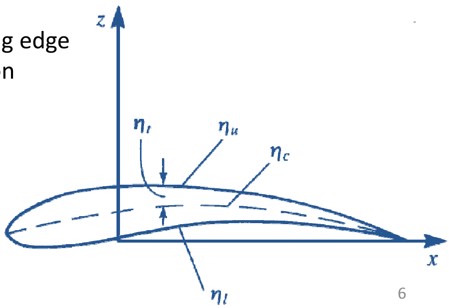
$$\frac{|\partial\Phi/\partial x|}{Q_\infty}, \frac{|\partial\Phi/\partial y|}{Q_\infty}, \frac{|\partial\Phi/\partial z|}{Q_\infty} \ll 1 \quad (4.13)$$

From B.C. of Eq. (4.12), the following restrictions on the geometry will follow:

$$\left|\frac{\partial\eta}{\partial x}\right| \ll 1, \left|\frac{\partial\eta}{\partial y}\right| \ll 1, \quad \text{and} \quad \left|\frac{W_\infty}{U_\infty}\right| = \tan\alpha \approx \alpha \ll 1 \quad (4.14)$$

This means that the wing must be thin compared to its chord.

Note: near stagnation points and near the leading edge (where ∂η/∂x is not small), the small perturbation assumption is not valid.



4.2 The Boundary Condition on the Wing

for small α $\begin{cases} W_\infty \approx Q_\infty \alpha \\ U_\infty \approx Q_\infty \end{cases}$ Eq. (4.12) reduce to

$$\frac{\partial \Phi}{\partial z}(x, y, \eta) = Q_\infty \left(\frac{\partial \eta}{\partial x} - \alpha \right) \quad (4.15)$$

Approximating B.C. from the wing surface to the *x*–*y* plane by a Taylor series expansion:

$$\frac{\partial \Phi}{\partial z}(x, y, z = \eta) = \underbrace{\frac{\partial \Phi}{\partial z}(x, y, 0)}_{\text{only use}} + \eta \frac{\partial^2 \Phi}{\partial z^2}(x, y, 0) + O(\eta^2) \quad (4.16)$$

only use

The first-order approximation of B.C. Eq. (4.12)

$$\frac{\partial \Phi}{\partial z}(x, y, 0) = Q_\infty \left(\frac{\partial \eta}{\partial x} - \alpha \right) \quad (4.17)$$

linear B.C. defined for a thin wing

A higher order approximation will be considered in Chapter 7

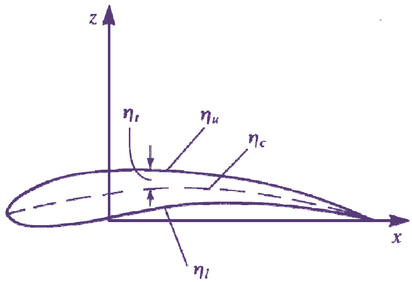
4.3 Separation of the Thickness and the Lifting Problems

The shape of the wing is then defined by:

$$z = \eta_u(x, y)$$
$$z = \eta_l(x, y)$$
$$\eta_c = \frac{1}{2}(\eta_u + \eta_l)$$
$$\eta_t = \frac{1}{2}(\eta_u - \eta_l)$$

$\rightarrow$

$$\eta_u = \eta_c + \eta_t$$
$$\eta_l = \eta_c - \eta_t$$



Spacifing linear B.C. Eq. (4.17) for both upper & lower wing surfaces

$$\frac{\partial \Phi}{\partial z}(x, y, 0+) = \left(\frac{\partial \eta_c}{\partial x} + \frac{\partial \eta_t}{\partial x} \right) Q_\infty - Q_\infty \alpha \quad (4.21a)$$

$$\frac{\partial \Phi}{\partial z}(x, y, 0-) = \left(\frac{\partial \eta_c}{\partial x} - \frac{\partial \eta_t}{\partial x} \right) Q_\infty - Q_\infty \alpha \quad (4.21b)$$

$\leftarrow \frac{\partial \Phi}{\partial z}(x, y, 0) = Q_\infty \left(\frac{\partial \eta}{\partial x} - \alpha \right)$

B.C. at infinity (Eq. (4.2)), for the perturbation potential

$$\lim_{r \to \infty} \nabla \Phi = 0 \quad (4.21c)$$

4.3 Separation of the Thickness and the Lifting Problems

Summery: Thin wing continuity equation & its B.C.

$$\nabla^2 \Phi = 0 \quad (4.11)$$

B.C.

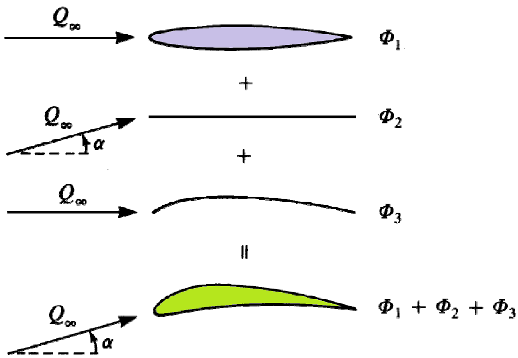
$$\frac{\partial \Phi}{\partial z}(x, y, 0+) = \left(\frac{\partial \eta_c}{\partial x} + \frac{\partial \eta_t}{\partial x}\right) Q_\infty - Q_\infty \alpha \quad (4.21a)$$

$$\frac{\partial \Phi}{\partial z}(x, y, 0-) = \left(\frac{\partial \eta_c}{\partial x} - \frac{\partial \eta_t}{\partial x}\right) Q_\infty - Q_\infty \alpha \quad (4.21b)$$

$$\lim_{r \to \infty} \nabla \Phi = 0 \quad (4.21c)$$

All of above equations are linear

it is possible to solve three simpler problems and superimpose the three separate solutions.



4.3 Separation of the Thickness and the Lifting Problems

1. Symmetric wing with nonzero thickness at zero angle of attack (effect of thickness):

$$\nabla^2 \Phi_1 = 0 \quad (4.22)$$

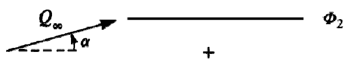
$$\frac{\partial \Phi_1}{\partial z}(x, y, 0\pm) = \pm \frac{\partial \eta_t}{\partial x} Q_\infty \quad (4.23)$$

where + is for the upper and – is for the lower surfaces.

2. Zero-thickness, uncambered wing at angle of attack (effect of angle of attack):

$$\nabla^2 \Phi_2 = 0 \quad (4.24)$$

$$\frac{\partial \Phi_2}{\partial z}(x, y, 0\pm) = -Q_\infty \alpha \quad (4.25)$$



3. Zero-thickness, cambered wing at zero angle of attack (effect of camber):

$$\nabla^2 \Phi_3 = 0 \quad (4.26)$$

$$\frac{\partial \Phi_3}{\partial z}(x, y, 0\pm) = \frac{\partial \eta_c}{\partial x} Q_\infty \quad (4.27)$$

The complete solution for the cambered wing with nonzero thickness at an angle of attack

$$\Phi = \Phi_1 + \Phi_2 + \Phi_3 \quad (4.28)$$

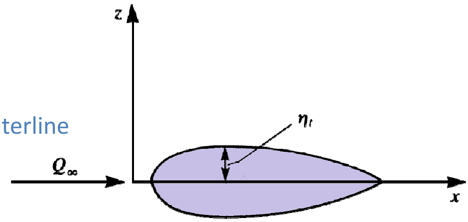
all three linear B.C.have to be fulfilled at wing's projected area on the z = 0 plane

4.4 Symmetric Wing with Nonzero Thickness at Zero AOA

A symmetric wing with a thickness distribution of $\eta_t\ (x, y)$

$$\nabla^2 \Phi = 0 \qquad (4.29)$$

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = \pm \frac{\partial \eta_t}{\partial x} Q_\infty \qquad (4.30)$$



Symmetry → source/sink → at wing section centerline

The potential of point source element σ

$$\Phi = \frac{-\sigma}{4\pi r}$$

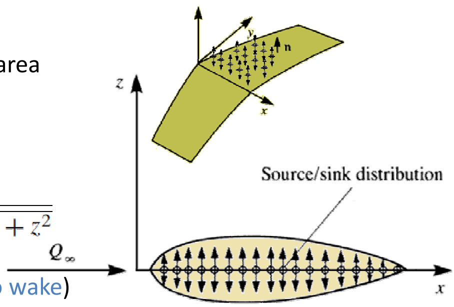
where r is the distance from the point singularity located at (x_0, y_0, z_0)

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

elements are distributed over wing's projected area on the x–y plane (z_0 = 0)

The velocity potential at point (x, y, z)

$$\Phi(x, y, z) = \frac{-1}{4\pi} \int_{\text{wing}} \frac{\sigma(x_0, y_0)\, dx_0\, dy_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}}$$



Note: Integration is done over the wing only (no wake)

4.4 Symmetric Wing with Nonzero Thickness at Zero AOA

The normal velocity component w(x, y, z)

$$w(x, y, z) = \frac{\partial \Phi}{\partial z} = \frac{z}{4\pi} \int_{\text{wing}} \frac{\sigma(x_0, y_0)\, dx_0\, dy_0}{[(x - x_0)^2 + (y - y_0)^2 + z^2]^{3/2}}$$

$$w(x, y, 0\pm) = \lim_{z \to 0\pm} w(x, y, z) = \pm \frac{\sigma(x, y)}{2} \qquad (4.35)$$ ← From Chapter 3

OR

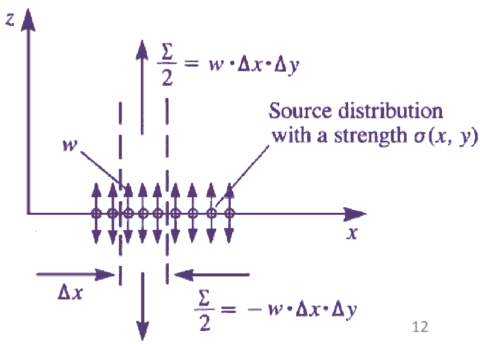
obtaining by observing the volume flow rate

$\Sigma = \sigma(x, y) \Delta x \Delta y$ volumetric flow

$dz \to 0$

$$\Sigma = 2w(x, y, 0+) \Delta x \Delta y = \sigma(x, y) \Delta x \Delta y$$

$$w(x, y, 0\pm) = \pm \frac{\sigma(x, y)}{2} \qquad (4.35)$$



4.4 Symmetric Wing with Nonzero Thickness at Zero AOA

Substitution of Eq. (4.35) into the boundary condition

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = \pm\frac{\partial \eta_t}{\partial x} Q_\infty = \pm\frac{\sigma(x, y)}{2} \rightarrow \boxed{\sigma(x, y) = 2Q_\infty \frac{\partial \eta_t}{\partial x}(x, y)}$$

The source distribution is easily obtained

$$\Phi(x, y, z) = \frac{-1}{4\pi} \int_{\text{wing}} \frac{\sigma(x_0, y_0)\, dx_0\, dy_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}}$$

The velocity potential and differentiating to obtain the velocity field

$$\Phi(x, y, z) = \frac{-Q_\infty}{2\pi} \int_{\text{wing}} \frac{[\partial \eta_t(x_0, y_0)/\partial x]\, dx_0\, dy_0}{\sqrt{(x - x_0)^2 + (y - y_0)^2 + z^2}} \qquad (4.37)$$

$$u(x, y, z) = \frac{Q_\infty}{2\pi} \int_{\text{wing}} \frac{[\partial \eta_t(x_0, y_0)/\partial x](x - x_0)\, dx_0\, dy_0}{[(x - x_0)^2 + (y - y_0) + z^2]^{3/2}} \qquad (4.38)$$

$$v(x, y, z) = \frac{Q_\infty}{2\pi} \int_{\text{wing}} \frac{[\partial \eta_t(x_0, y_0)/\partial x](y - y_0)\, dx_0\, dy_0}{[(x - x_0)^2 + (y - y_0) + z^2]^{3/2}} \qquad (4.39)$$

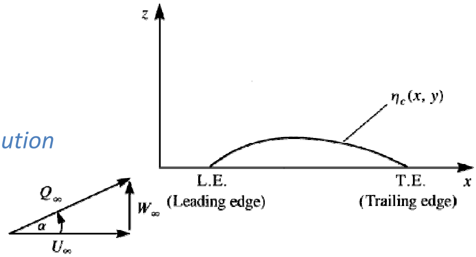
$$w(x, y, z) = \frac{Q_\infty}{2\pi} \int_{\text{wing}} \frac{[\partial \eta_t(x_0, y_0)/\partial x] z\, dx_0\, dy_0}{[(x - x_0)^2 + (y - y_0) + z^2]^{3/2}} \qquad (4.40)$$

4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

Solving the two linear problems of angle of attack and camber together

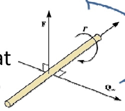
$$\nabla^2 \Phi = 0 \qquad (4.29)$$

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = Q_\infty \left(\frac{\partial \eta_c}{\partial x} - \alpha \right) \qquad (4.41)$$



Antisymmetric Problem respect to the z direction → *doublet distribution* / *vortex distribution*

if the lifting problem is to be modeled with vortex elements they cannot be terminated at the wing and must be shed into the flow. So as not to generate force in the fluid, these free vortex elements must be parallel to the local flow direction, at any point on the wake



$$\mathbf{Q} \times \mathbf{\Gamma}$$

Note: for small-disturbance approximation, wake should be planar and placed on z = 0 plane

4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

a. Doublet Distribution

The doublets pointing in the z direction that create a pressure jump in this direction.
Velocity Potential of Doublet (antisymmetric point element) placed at (x_0, y_0, z_0)

$$\Phi(x, y, z) = \frac{-\mu(x_0, y_0)(z - z_0)}{4\pi[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}$$

Potential at an arbitrary point (x, y, z) due to these elements distributed over the wing and its wake, ($z_0 = 0$)

$$\Phi(x, y, z) = \frac{1}{4\pi}\int_{\text{wing+wake}} \frac{-\mu(x_0, y_0) z \, dx_0 \, dy_0}{[(x - x_0)^2 + (y - y_0)^2 + z^2]^{3/2}}$$

The velocity is obtained by differentiating above Eq. and letting z → 0 on the wing

$$u(x, y, 0\pm) = \frac{\partial \Phi}{\partial x} = \frac{\mp 1}{2}\frac{\partial \mu}{\partial x}$$

$$v(x, y, 0\pm) = \frac{\partial \Phi}{\partial y} = \frac{\mp 1}{2}\frac{\partial \mu}{\partial y}$$

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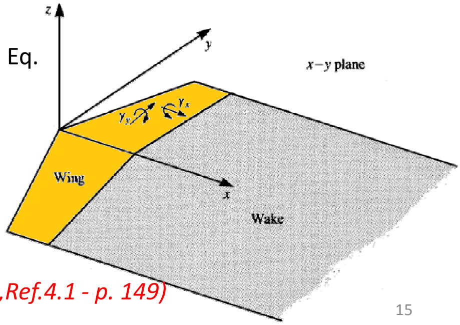
$$w(x, y, 0\pm) = \frac{\partial \Phi}{\partial z}$$

(see Ashley and Landahl,Ref.4.1 - p. 149)



4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

From Ashley and Landahl, (Ref. 4.1):

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = \frac{1}{4\pi}\int_{\text{wing+wake}} \frac{\mu(x_0, y_0)}{(y - y_0)^2}\left[1 + \frac{(x - x_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}\right] dx_0 \, dy_0 \quad (4.44)$$

B.C. Eq.(4.41) = Eq. (4.44)

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = Q_\infty\left(\frac{\partial \eta_c}{\partial x} - \alpha\right)$$

$$Q_\infty\left(\frac{\partial \eta_c}{\partial x} - \alpha\right) = \frac{1}{4\pi}\int_{\text{wing+wake}} \frac{\mu(x_0, y_0)}{(y - y_0)^2}\left[1 + \frac{(x - x_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}\right] dx_0 \, dy_0$$

The integral equation for the unknown *μ(x, y)*

4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

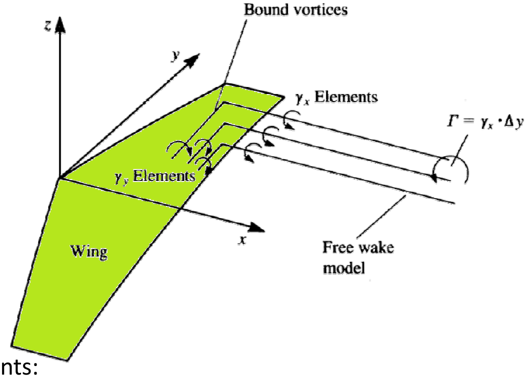
b. Vortex Distribution

Vortex line distributes over the wing and wake, as in the case of doublet distribution. Computing the velocity Δ**q** due a vortex line element d**l** with a strength of ΔΓ by the Biot–Savart law

$$\Delta \mathbf{q} = \frac{-1}{4\pi} \frac{\Delta\Gamma \mathbf{r} \times d\mathbf{l}}{r^3}$$

γ_y :*Vortex element point in y-direction*

γ_x :*Vortex element point in x-direction*



The component of velocity normal to the wing (downwash), induced by these elements:

$$w(x, y, z) = \frac{-1}{4\pi} \int_{\text{wing+wake}} \frac{\gamma_y(x - x_0) - \gamma_x(y - y_0)}{r^3} \, dx_0 \, dy_0 \qquad (4.46)$$

In this formulation there are two unknown quantities per point (γ_x, γ_y)

From Helmholtz vortex theorems: vortex strength is constant along a vortex line

4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

Considering the vortex distribution on the wing to consist of a large number of infinitesimal vortex lines, then at any point on the wing

$$|\partial \gamma_x / \partial x| = |\partial \gamma_y / \partial y|$$

The final number of unknowns at a point is reduced to one.

For a vortex distribution

$$u(x, y, 0\pm) = \frac{\partial \Phi}{\partial x} = \frac{\pm\gamma_y(x, y)}{2}$$

$$v(x, y, 0\pm) = \frac{\partial \Phi}{\partial y} = \frac{\mp\gamma_x(x, y)}{2}$$

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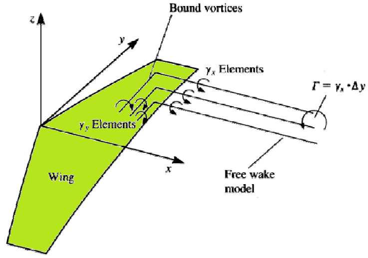
Obtaining the velocity potential on the wing at any point x ($y = y_0$ = const.) by integrating the x component of the velocity along an x-wise line beginning at the leading edge (L.E.):

$$\Phi(x, y_0, 0\pm) = \int_{L.E.}^{x} u(x_1, y_0, 0\pm)\, dx_1 \quad \textit{OR} \quad \Delta\Phi(x, y_0) = \int_{L.E.}^{x} \gamma_y(x_1, y_0)\, dx_1$$

4.5 Zero-Thickness Cambered Wing at AOA–Lifting Surfaces

To construct the lifting surface equation for the unknown γ , the wing-induced downwash must be equal and opposite in sign to the normal component of the free-stream velocity:



$$w(x, y, z) = \frac{-1}{4\pi} \int_{\text{wing+wake}} \frac{\gamma_y(x - x_0) - \gamma_x(y - y_0)}{r^3}\, dx_0\, dy_0$$

$$\frac{\partial \Phi}{\partial z}(x, y, 0\pm) = Q_\infty \left(\frac{\partial \eta_c}{\partial x} - \alpha \right)$$

B.C. Eq.(4.41) = Eq. (4.46)

⇩

$$\frac{-1}{4\pi} \int_{\text{wing+wake}} \frac{\gamma_y(x - x_0) - \gamma_x(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^{3/2}}\, dx_0\, dy_0 = Q_\infty \left(\frac{\partial \eta_c}{\partial x} - \alpha \right)$$

The integral equation for the unknown *γ(x, y)*

4.6 The Aerodynamic Loads

Thickness or Lifting problems ⇨ Velocity Field ⇨ Pressures Distribution ⇨ Aerodynamic Loads

The velocity at any point in the field = free-stream velocity + Perturbation velocity

$$\mathbf{q} = \left(Q_\infty \cos\alpha + \frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z} + Q_\infty \sin\alpha \right)$$

Substituting **q** into → Bernoulli equation + Small Disturbance Assumptions (Eqs. (4.13) & (4.14) & α << 1):

$$p_\infty - p = \frac{\rho}{2}(q^2 - Q_\infty^2)$$

$$p_\infty - p = \frac{\rho}{2}\left[Q_\infty^2 \cos^2\alpha + 2Q_\infty \cos\alpha \frac{\partial \Phi}{\partial x} + \left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2 + \left(Q_\infty \sin\alpha + \frac{\partial \Phi}{\partial z}\right)^2 - Q_\infty^2 \right]$$

$$p_\infty - p = \rho Q_\infty \frac{\partial \Phi}{\partial x}$$

The pressure coefficient:

$$C_p \equiv \frac{p - p_\infty}{(1/2)\rho Q_\infty^2} = 1 - \left(\frac{q}{Q_\infty}\right)^2 = -2\frac{\partial \Phi / \partial x}{Q_\infty} \qquad (4.53)$$

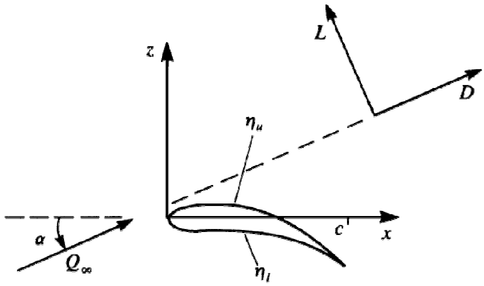
4.6 The Aerodynamic Loads

The aerodynamic loads:

$$\mathbf{F} = -\int_{\text{wing}} p\mathbf{n}\, dS$$

The normal to the surface with
the small-disturbance approximation:

$$\mathbf{n} = \frac{1}{|\nabla F|}\left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1\right) \approx \left(-\frac{\partial \eta}{\partial x}, -\frac{\partial \eta}{\partial y}, 1\right)$$



Wing-attached coordinate system

The components of the force **F:**

$$F_x = \int_{\text{wing}} \left(p_u \frac{\partial \eta_u}{\partial x} - p_l \frac{\partial \eta_l}{\partial x}\right) dx\, dy$$

$$F_y = \int_{\text{wing}} \left(p_u \frac{\partial \eta_u}{\partial y} - p_l \frac{\partial \eta_l}{\partial y}\right) dx\, dy$$

$$F_z = \int_{\text{wing}} (p_l - p_u)\, dx\, dy$$

⇨

$$D = F_x \cos\alpha + F_z \sin\alpha$$

$$L = -F_x \sin\alpha + F_z \cos\alpha \approx F_z$$

4.6 The Aerodynamic Loads

Evaluating pressure difference across the thin wing (Δp)
positive Δp is in the +z direction

$$\Delta p = p_l - p_u = p_\infty - \rho Q_\infty \frac{\partial \Phi}{\partial x}(x, y, 0-) - \left[p_\infty - \rho Q_\infty \frac{\partial \Phi}{\partial x}(x, y, 0+)\right]$$

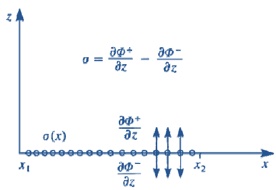
$$\Delta p = \rho Q_\infty \left[\frac{\partial \Phi}{\partial x}(x, y, 0+) - \frac{\partial \Phi}{\partial x}(x, y, 0-)\right]$$

Pressure Differences → The Singularity Distribution placed on the x–y plane → Source / Doublet / Vortex

1. Source distribution:

Symmetry → $\frac{\partial \Phi}{\partial x}(x, y, 0+) = \frac{\partial \Phi}{\partial x}(x, y, 0-)$

$$\Delta p = \rho Q_\infty \left[\frac{\partial \Phi}{\partial x}(x, y, 0+) - \frac{\partial \Phi}{\partial x}(x, y, 0+)\right] = 0$$

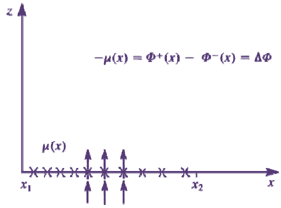


4.6 The Aerodynamic Loads

2. Doublet distribution:

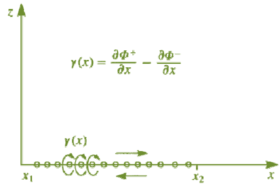
$$\frac{\partial \Phi}{\partial x}(x, y, 0\pm) = \frac{\mp 1}{2}\frac{\partial \mu(x, y)}{\partial x}$$

$$\Delta p = \rho Q_\infty \frac{\partial}{\partial x}\Delta\Phi(x, y) = -\rho Q_\infty \frac{\partial \mu(x, y)}{\partial x}$$



3. Vortex distribution:

For the vortex distribution on the x–y plane the pressure jump can be modeled with a vortex distribution $\gamma_y(x, y)$



$$\frac{\partial \Phi}{\partial x}(x, y, 0\pm) = \frac{\pm\gamma_y(x, y)}{2}$$

$$\Delta p = \rho Q_\infty \frac{\partial}{\partial x}\Delta\Phi(x, y) = \rho Q_\infty \gamma_y(x, y)$$

Pitching moment about y axis for a wing placed at z = 0 surface

$$M_{x=0} = -\int_{\text{wing}} \Delta p x \, dx \, dy$$

$$\begin{cases} C_M = \dfrac{M}{(1/2)\rho Q_\infty^2 S b} \\ C_F = \dfrac{F}{(1/2)\rho Q_\infty^2 S} \end{cases}$$

S: Refrence area (wing planform area)

b: Refrence moment arm (wing span)

4.7 The Vortex Wake

A combination of basic solution elements

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Satisfy zero normal flow B.C. on solid surfaces

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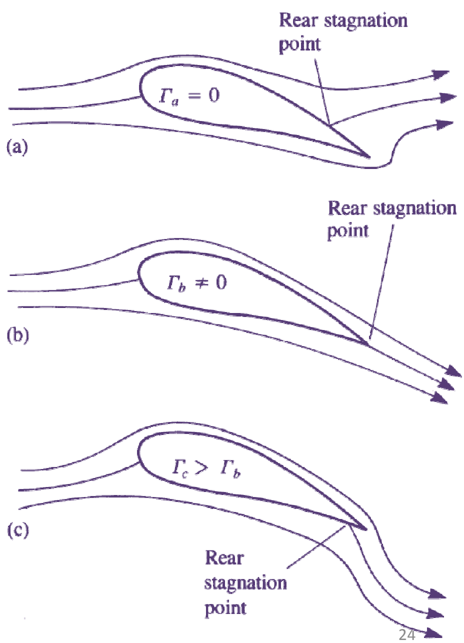
Solution is not unique (arbitrary value for Γ)



(a) The circulation is zero .
(b) The circulation is such that the flow at the trailing edge (T.E.) seems to be parallel at the edge.
(c) the circulation is even larger and the flow turns downward near the trailing edge (this can be achieved, for example, by blowing).

4.7 The Vortex Wake

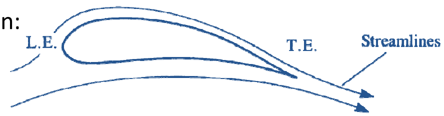
The *Kutta condition* states that:
The flow leaves the sharp trailing edge of an airfoil smoothly & the velocity there is finite.

Finite T.E. angle: normal component of the velocity, from both sides of the airfoil, must vanish. for a continuous velocity, this is possible only if this is a stagnation point. Therefore, it is useful to assume that the pressure difference there is also zero

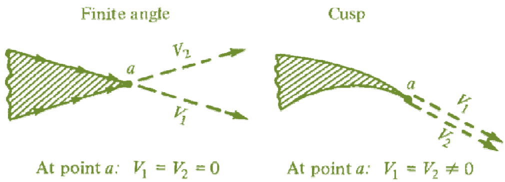
$$\Delta p_{T.E.} = 0 \qquad (4.63)$$

If the circulation is modeled by a vortex distribution:

$$\gamma_{T.E.} = 0 \longleftarrow \Delta p = \rho Q_\infty \gamma_y(x, y)$$



Cusped T.E.(zero angle): flow leaves T.E. along the bisector line smoothly with finite velocity.
Note: for Cusped T.E. Eq. (4.63) must hold even though the trailing edge need not be a stagnation point.



4.7 The Vortex Wake

Using vortex distribution to model the lift ⟶ Wing as the bound vortex $\gamma_y(x, y)$
Helmholtz's theorem

1. A vortex line cannot begin or end in the fluid
2. Any change in $\gamma_y(x, y)$ must be followed by an equal change in $\gamma_x(x, y)$.

The wing will be modeled by:

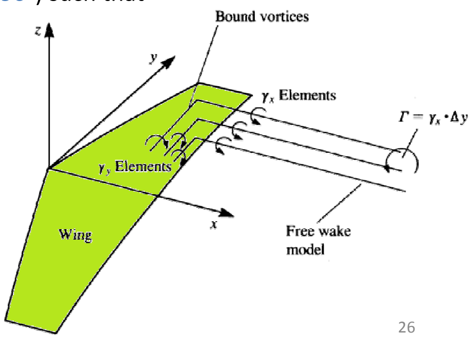
1. Constant-strength vortex lines,
2. If a change in the local strength of $\gamma_y(x, y)$ is needed then an additional vortex line will be added (or the vortex line is bent by ±90°) such that

$$\left|\frac{\partial \gamma_x(x, y)}{\partial x}\right| = \left|\frac{\partial \gamma_y(x, y)}{\partial y}\right|$$

Velocity induced by vortex distribution at a point slightly above (z = 0+) the wing:

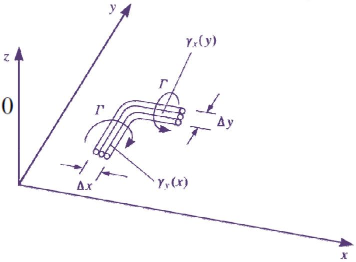
$$u(x, y, 0+) = \frac{\gamma_y(x, y)}{2}$$

$$v(x, y, 0+) = -\frac{\gamma_x(x, y)}{2}$$



4.7 The Vortex Wake

Vorticity free requires that:

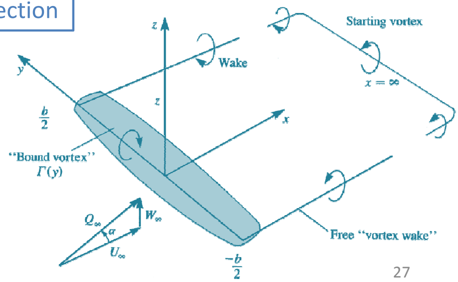


$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{4}\left(\frac{-\partial \gamma_x(x,y)}{\partial x} - \frac{\partial \gamma_y(x,y)}{\partial y}\right) = 0$$

$$\left|\frac{\partial \gamma_x(x,y)}{\partial x}\right| = \left|\frac{\partial \gamma_y(x,y)}{\partial y}\right|$$

Any change in vorticity in one direction must be followed by a change in a normal direction

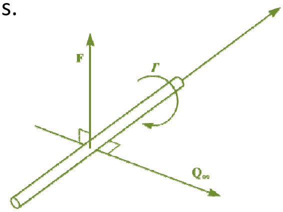
In the case of the wing the lifting vortices (bound vortices) cannot end at the wing (e.g., at the tip) and must be extended behind the wing into a wake. Furthermore, a lifting wing creates a starting vortex and this vortex may be located far downstream.



4.7 The Vortex Wake

If the wake is modeled by free vortex sheet, it is not creating loads.

$$\Delta p = \rho \mathbf{q} \times \gamma = 0 \qquad \textit{OR} \qquad \mathbf{q} \times \gamma = 0$$



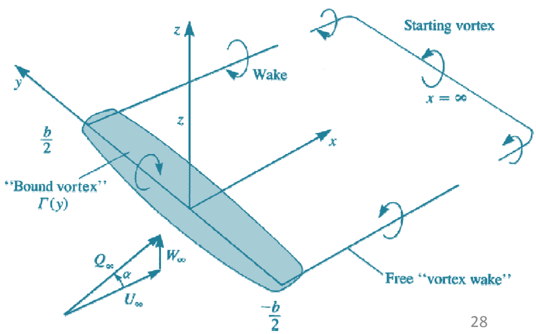
$$\mathbf{F} = \rho \mathbf{Q}_\infty \times \mathbf{\Gamma}$$

This means that the velocity on the wake must be parallel to the wake vortices.
A small-disturbance approximation applied to the wake model results in:

$$\mathbf{Q}_\infty \times \gamma_w = 0$$

$$\mathbf{Q}_\infty \parallel \gamma_w$$

Vortex lines in the wake are parallel to the free-stream.



4.8 Linearized Theory of Small-Disturbance Compressible Flow

Small disturbance assumption

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extending methods of incompressible potential flow to cover cases with small effects of compressibility (low-speed subsonic flows)

The continuity equation, Eq. (1.21) is rewritten in the form:

$$\frac{-1}{\rho}\left(\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

The inviscid momentum equations, Eqs. (1.31) are:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{-1}{\rho}\frac{\partial p}{\partial z}$$

The propagation speed of the disturbance a (speed of sound) in an isentropic fluid:

$$a^2=\frac{\partial p}{\partial \rho}$$

4.8 Linearized Theory of Small-Disturbance Compressible Flow

Replaced $\partial p/\partial x = a^2 \partial\rho/\partial x$, in the x direction & Multiplying the momentum equations by u, v, and w, respectively, and adding them together leads to

$$u\frac{\partial u}{\partial t}+v\frac{\partial v}{\partial t}+w\frac{\partial w}{\partial t}+u^2\frac{\partial u}{\partial x}+v^2\frac{\partial v}{\partial y}+w^2\frac{\partial w}{\partial z}+uv\frac{\partial u}{\partial y}+uv\frac{\partial v}{\partial x}+uw\frac{\partial u}{\partial z}$$
$$+uw\frac{\partial w}{\partial x}+vw\frac{\partial v}{\partial z}+vw\frac{\partial w}{\partial y}=\frac{-a^2}{\rho}\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}\right)$$

Replacing RHS with the continuity equation and recalling the irrotationality condition $\nabla \times \mathbf{q} = 0$

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x}+\left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y}+\left(1-\frac{w^2}{a^2}\right)\frac{\partial w}{\partial z}-2\frac{uv}{a^2}\frac{\partial u}{\partial y}-2\frac{vw}{a^2}\frac{\partial v}{\partial z}-2\frac{uw}{a^2}\frac{\partial w}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial \rho}{\partial t}-\frac{u}{a^2}\frac{\partial u}{\partial t}-\frac{v}{a^2}\frac{\partial v}{\partial t}-\frac{w}{a^2}\frac{\partial w}{\partial t}=0 \qquad (4.70)$$

Using the velocity potential & free-stream velocity $\mathbf{Q}_\infty = U_\infty \boldsymbol{i}$, and small disturbance assumption:

$$\left|\frac{\partial \Phi}{\partial x}\right|,\left|\frac{\partial \Phi}{\partial y}\right|,\left|\frac{\partial \Phi}{\partial z}\right| \ll U_\infty$$

$$\begin{cases} u = U_\infty + \dfrac{\partial \Phi}{\partial x} \\ v = \dfrac{\partial \Phi}{\partial y} \\ w = \dfrac{\partial \Phi}{\partial z} \end{cases}$$

4.8 Linearized Theory of Small-Disturbance Compressible Flow

Assuming steady-state flow (∂/∂t = 0), and neglecting the smaller terms in Eq. (4.70):

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

Using the energy equation for an adiabatic flow, we can show that the local speed of sound can be replaced by its free-stream value and the small-disturbance equation becomes:

$$(1-M_\infty^2)\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0$$

Using a simple coordinate transformation , called the Prandtl–Glauert rule*:*

$$x_M=\frac{x}{\sqrt{1-M_\infty^2}}$$

$$y_M=y \quad\rightarrow\quad \partial/\partial x_M=(1-M_\infty^2)^{-1/2}\partial/\partial x$$

$$z_M=z$$

4.8 Linearized Theory of Small-Disturbance Compressible Flow

The pressure coefficient of Eq. (4.53) becomes:

$$C_p=-2\frac{\partial\Phi/\partial x_M}{Q_\infty}=-2\frac{\partial\Phi/\partial x}{Q_\infty}\frac{1}{\sqrt{1-M_\infty^2}}$$

Similarly the lift and moment coefficients become:

$$C_L(M>0)=\frac{C_L(M=0)}{\sqrt{1-M_\infty^2}}$$

$$C_M(M>0)=\frac{C_M(M=0)}{\sqrt{1-M_\infty^2}}$$

which indicates that at higher speeds the lift slope is increasing.

Applicable at least up to $M_\infty=0.5$.